WALTER EVANS' PINBALL METHOD

It is a pleasure to send you an example of use of the pinball method. (I don't happen to remember it by that name; but memories are funny.) Oh! I just got your latest; and maybe I'm remembering a different thing. Oh well, having spent two days on it, I might as well send it now. Hope it's helpful.

There was a delightful period in my career when I shared Walt's office with him and Bill Mullins. They were two very different, very smart and creative gents. Walt already had in his small office -- besides his desk -- a large ordinary table with a large paper pad on it (which was known as "the genius pad"). There were always smart people on the other side of table from him, and a lively technical argument in progress. (Walt was always right.)

One day Walt decided he would enjoy having me and Bill in the same office with him. We both jumped at the chance; and Walt had two more desks moved in. We were all solving problems all day, often with consultation from each other. Consultation from Walt was like nothing else. The "pinball method" is a minor example.

The easy way for me to tell you about the method I remember is to use it to solve a simple, typical problem. (This is how I always teach things.) You will then find it easy to generalize use of the method to any polynomial whose roots you would like to know.

But you do have to put it in historical context: There were at that time no digital computers. At all. The spirule is an ingenious analog substitute (by Evans) with which we were designing control systems full bore at that time (e.g., for inertial navigation stable platforms). It went with Walt's invention of what he called the Root Locus method and which I renamed the Evans Method (when I wrote a book called "Dynamics of Physical Systems"). I also defined the Evans Function as an easy way to formulate the process of finding the roots of the characteristic equation of any linear system (with constant coefficients), which is the core of designing a control system. The pinball method is a simple special-case use of the Evans Root Locus Method.

Suppose you have the polynomial[1]

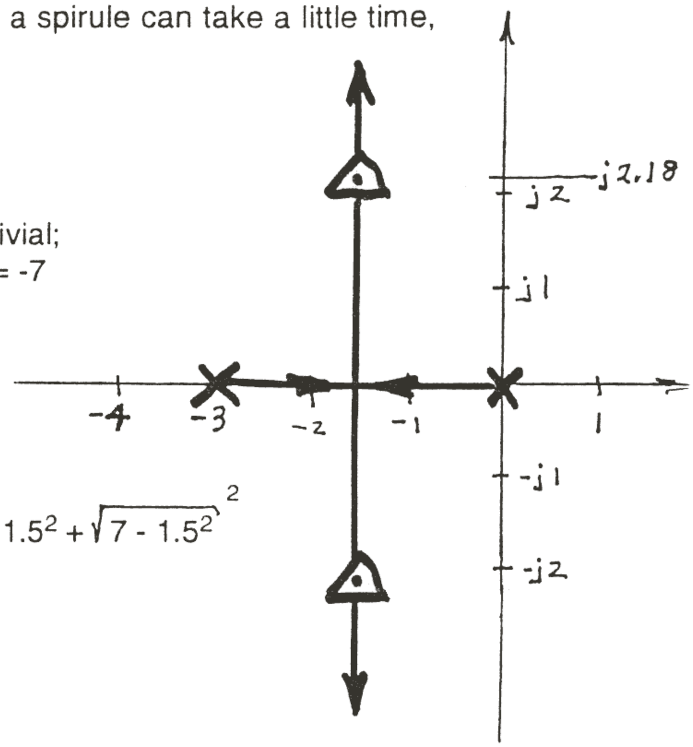
$$s^5 + 3s^4 + 7s^3 + 10s^2 + 18s + 80 = 0$$

You simply rewrite it in the form:

$$s\left\{s\left[s\left[s\,(s+3)+7\right]+10\right]+18\right\}+80=0$$

[1] I've used s because (a) I'm used to working with polynomials that are the characteristic equations of dynamic systems, derived by assuming a solution to its differential equations of the form $x = Ce^{st}$; and (b) because it takes only one stroke to make an s.

Then you factor each bracketed term in turn, starting with the inside one, and using the Root Locus Method for each successive factoring, as we'll now demonstrate. [To make the historical point, I did all the root locus plots in this little treatise with a spirule -- only! The accuracy is about two and a half significant figures. Today you would have a simple digital computer program to plot them for you in seconds with however many significant figures you wanted. In real life your data are typically good to only two or three significant figures anyway; but plotting with a spirule can take a little time, compared with a digital computer.]



Step 1. Factoring the first bracket is of course trivial; but let's plot it's locus of roots anyway, $s\ (s + 3) = -7$ and then check the result algebraically:

Roots: $s = -1.5 + j2.18$ $\qquad s = -1.5 - j2.8$

or $\quad (s + 1.5 + j\sqrt{7 - 1.5^2})\ (s + 1.5 - j\sqrt{7 - 1.5^2})$

$$s^2 + (1.5 + j\sqrt{7 - 1.5^2} + 1.5 - j\sqrt{7 - 1.5^2})s + 1.5^2 + \sqrt{7 - 1.5^2}^{\,2}$$

$$s^2 + 3s + 7$$

Step 2. Factoring the second bracket is a classical third-order locus plot. It is given on the page labeled "Step 2" herewith. The roots are found to be:

$s = -2 \qquad s = -.5 + j2.18 \qquad s = -.5 - j2.18$

which of course means that the second bracket, in factored form, is:

$$(s + 2)\ (s + .5 \pm j2.18)$$

Again, it's easy to check by multiplying the factors out:

$$(s + 2)\ (s^2 + s + 5.0) = s^3 + 3s^2 + 7.0s + 10$$

Step 3. Factoring the third bracket is a fourth-order locus plot, which is a little more complex, as the page labeled "Step 3" shows. The resulting set of roots are:

$s = (-1.75 + j1.45)$ $s = -1.75 - j1.45$ $s = +.2 + j1.85$ $s = +.2 - j1.85$

So that the third bracket, in factored form, is:

$$(s^2 + 3.5s + 5.3)(s^2 - .4s + 3.46)$$

When I multiply these factors out I begin to see a bit of error creeping in:

$$s^4 + 3.1s^3 + 7.4s^2 + 9.8s + 18.3$$

Step 4. Factoring the fourth bracket is shown on the page labeled "Step 4." This is a fifth-order locus plot. The resulting roots are (to about two and one-half significant figures):

$s = -2.66$ $s = -1.3 + j2.5$ $s = -1.3 - j2.5$ $s = +1.1 + j1.7$ $s = +1.1 - j1.7$

Again as a check, we simply multiply the factored terms together, to obtain:

$$s^5 + 3.06s^4 + 6.8s^3 + 9.8s^2 + 15.9s + 80$$

Which seems to agree with the original polynomial to about two and on-half significant figures. I hope this has been helpful. (But maybe it has been irrelevant.)

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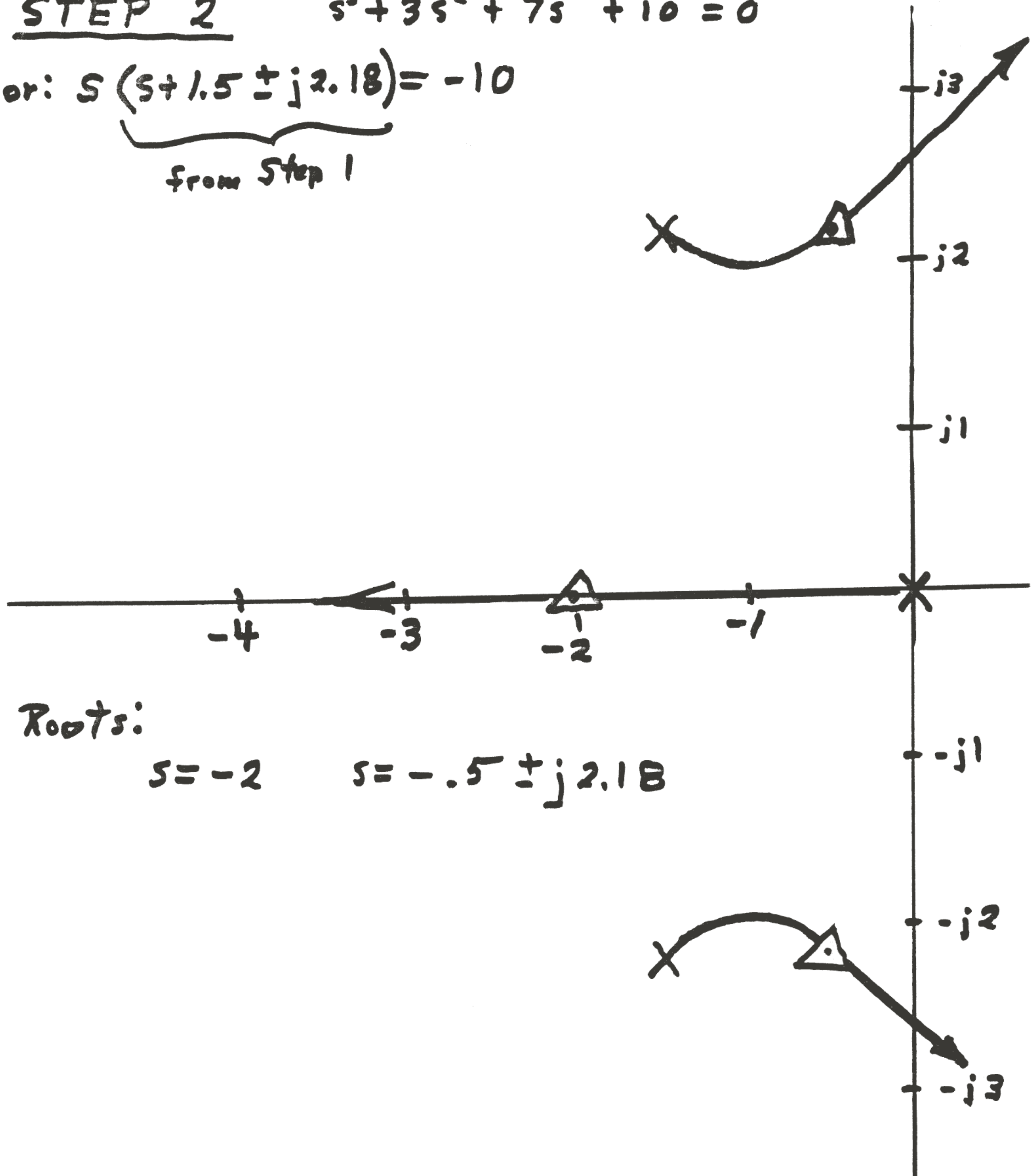
STEP 2

$$s^3 + 3s^2 + 7s + 10 = 0$$

or: $$s\underbrace{(s + 1.5 \pm j2.18)}_{\text{from Step 1}} = -10$$

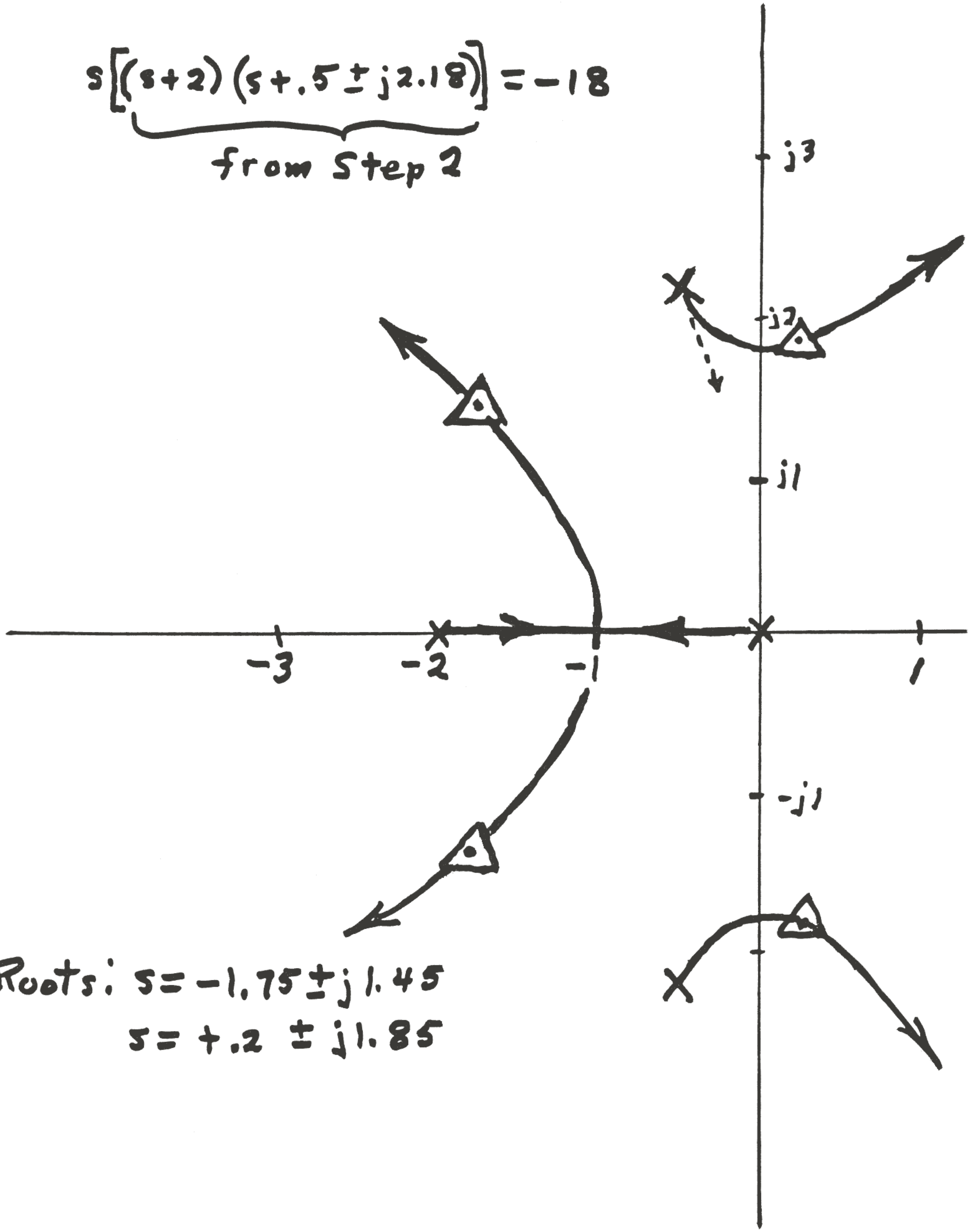


Roots:

$s = -2 \qquad s = -.5 \pm j2.18$

STEP 3

$$s\left[\underbrace{(s+2)\,(s+.5\pm j2.18)}_{\text{from Step 2}}\right]=-18$$

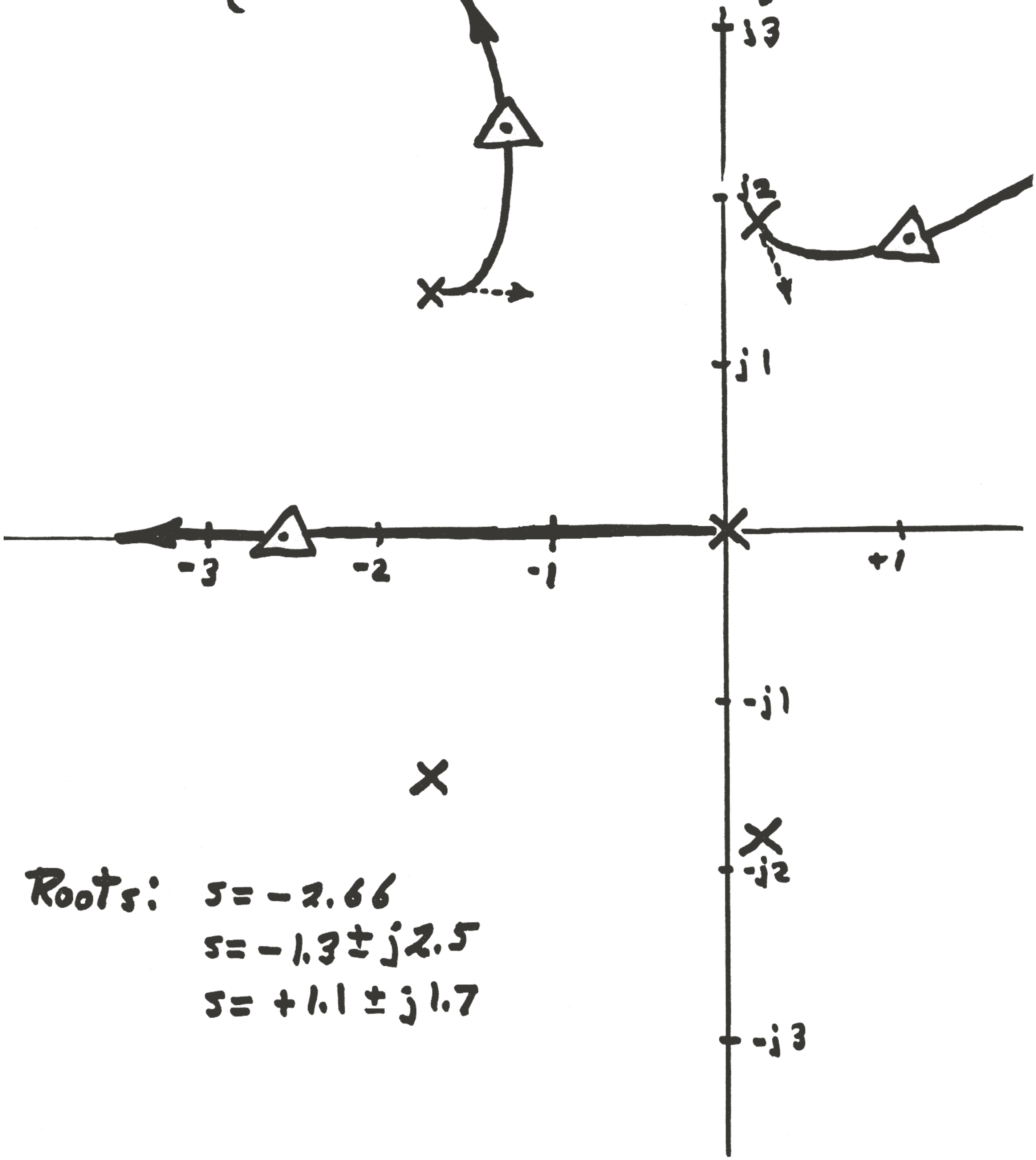


Roots: $s=-1.75\pm j1.45$

$s=+.2\pm j1.85$

STEP 4

$$s\left\{\overbrace{(s+1.75\pm j1.45)(s-.2\pm j1.85)}^{\text{from Step 3}}\right\}+80=0$$



Roots: $s=-2.66$

$s=-1.3\pm j2.5$

$s=+1.1\pm j1.7$